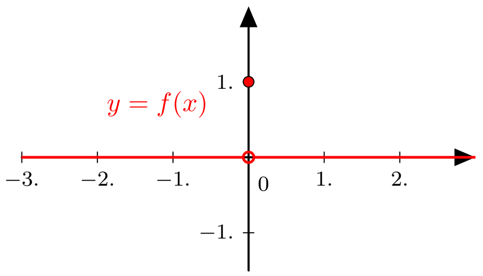
THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Suggested Solution to Problem Set 1

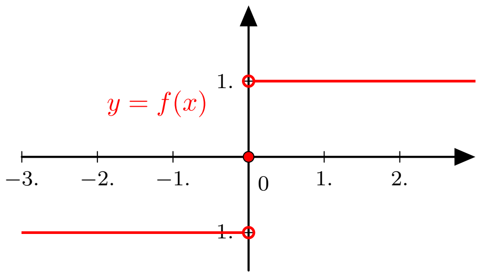
1. (a) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0. \end{cases}$$

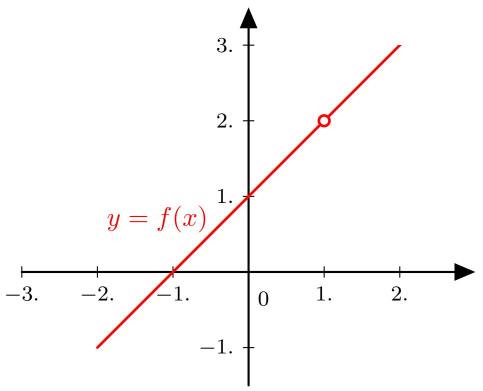



(b) $f : \mathbb{R} \to \mathbb{R}$ defined by

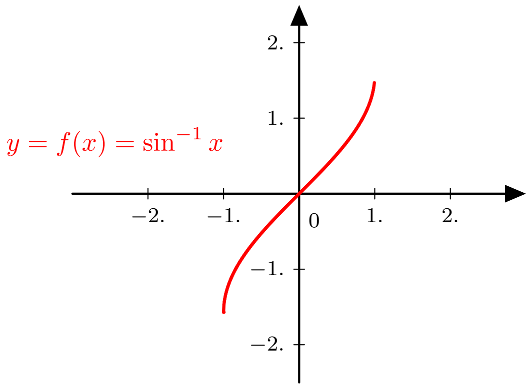
$$f(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$



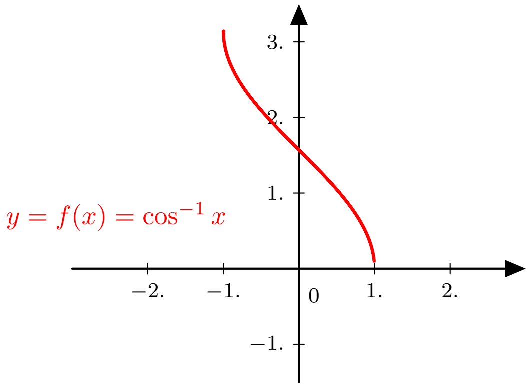

(c) $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$.



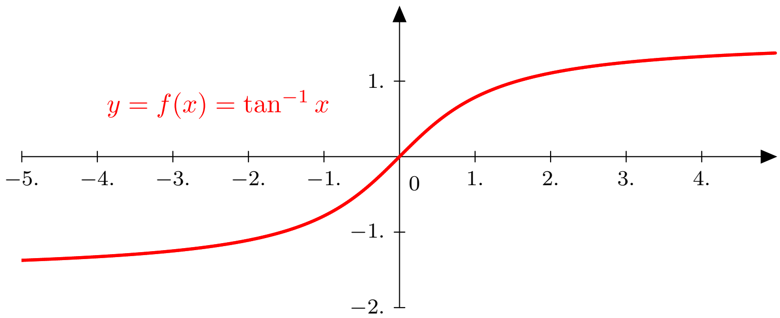

(d) $f : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by $f(x) = \sin^{-1} x$.




(e) $f : [-1, 1] \to [0, \pi]$ defined by $f(x) = \cos^{-1} x$.




(f) $f : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by $f(x) = \tan^{-1} x$.




2. Prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$ for all natural numbers $n$.

*Proof.* Let $P(n)$ be the statement that "$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$".

- When $n = 1$, L.H.S. $= 1 \times 2 = 2$ and R.H.S $= \frac{(1)(2)(3)}{3} = 2$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n$, i.e.

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}.$$

Then,

$$
\begin{aligned}
& 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) + (n+1) \times (n+2) \\
= & \frac{n(n+1)(n+2)}{3} + (n+1) \times (n+2) \\
= & \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3} \\
= & \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}
$$

Therefore, $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers $n$. □

3. Prove that $8^n - 3^n$ is divisible by 5 for all natural numbers $n$.

*Proof.* Let $P(n)$ be the statement that ”$8^n - 3^n$ is divisible by 5”.

- When $n = 1$, $8^1 - 3^1 = 5$ which is divisible by 5. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n$, i.e. $8^n - 3^n = 5M$ for some integer $M$.

  Then,

$$
\begin{aligned}
& 8^{n+1} - 3^{n+1} \\
= & 8 \times 3^n - 3 \times 3^n \\
= & 8 \times (8^n - 3^n) + 5 \times 3^n \\
= & 8 \times (5M) + 5 \times 3^n \\
= & 5 \times (8M + 3^n)
\end{aligned}
$$

  where $8M + 3^n$ is an integer. Therefore, $8^{n+1} - 3^{n+1}$ is divisible by 5 and $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers $n$. □

4. A sequence $\{a_n\}$ is defined by

$$a_1 = 3 \text{ and } a_{n+1} = \sqrt{a_n + 5} \text{ for } n \geq 1.$$

Show that $a_n \geq a_{n+1}$ for all natural numbers $n$.

*Proof.* Let $P(n)$ be the statement that ”$a_n \geq a_{n+1}$”.

- When $n = 1$, $a_2 = \sqrt{a_1 + 5} = \sqrt{8} \leq \sqrt{9} = 3 = a_1$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n$, i.e. $a_n \geq a_{n+1}$.

  Then,

$$
\begin{aligned}
a_n & \geq a_{n+1} \\
a_n + 5 & \geq a_{n+1} + 5 \\
\sqrt{a_n + 5} & \geq \sqrt{a_{n+1} + 5} \\
a_{n+1} & \geq a_{n+2}
\end{aligned}
$$

  Therefore, $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers $n$. $\square$

5. A sequence $\{a_n\}$ is defined by

$$a_1 = 4 \text{ and } a_{n+1} = \frac{6(a_n^2+1)}{a_n^2+11} \text{ for } n \geq 1.$$

(a) Show that $a_n > 3$ for all natural numbers $n$.

*Proof.* Let $P(n)$ be the statement that "$a_n > 3$".

- When $n = 1$, $a_1 = 4 > 3$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n$, i.e. $a_n > 3$.
Then,

$$\begin{aligned} a_{n+1} - 3 &= \frac{6(a_n^2+1)}{a_n^2+11} - 3 \\ &= \frac{3a_n^2 - 27}{a_n^2+11} \\ &= \frac{3(a_n-3)(a_n+3)}{a_n^2+11} \\ &> 0 \end{aligned}$$

Therefore, $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers $n$. $\square$

(b) Show that $a_n \geq a_{n+1}$ for all natural numbers $n$.

*Proof.* Let $P(n)$ be the statement that "$a_n \geq a_{n+1}$".

- When $n = 1$, $a_2 = \frac{6(a_1^2+1)}{a_1^2+11} = \frac{6(4^2+1)}{4^2+11} = \frac{34}{9} < 4 = a_1$. Therefore, $P(1)$ is true.
- Suppose $P(n)$ is true for some natural number $n$, i.e. $a_n \geq a_{n+1}$.
Then,

$$\begin{aligned} a_{n+1} - a_n &= \frac{6(a_n^2+1)}{a_n^2+11} - a_n \\ &= \frac{-a_n^3 + 6a_n^2 - 11a_n + 6}{a_n^2+11} \\ &= \frac{-(a_n-1)(a_n-2)(a_n-3)}{a_n^2+11} \\ &< 0 \qquad (\because a_n > 3) \end{aligned}$$

Therefore, $P(n+1)$ is true.

By mathematical induction, $P(n)$ is true for all natural numbers $n$. $\square$

6.

$$\begin{aligned}
\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ &= \frac{1}{2}(\cos 60^\circ + \cos 20^\circ)\cos 80^\circ \\
&= \frac{1}{2}(\frac{1}{2}\cos 80^\circ + \cos 20^\circ \cos 80^\circ) \\
&= \frac{1}{2}\Big(\frac{1}{2}\cos 80^\circ + \frac{1}{2}(\cos 100^\circ + \cos 60^\circ)\Big) \\
&= \frac{1}{2}\Big(\frac{1}{2}\cos 80^\circ + \frac{1}{2}\cos 100^\circ + \frac{1}{4}\Big) \\
&= \frac{1}{8}
\end{aligned}$$

7.

$$\begin{aligned}
\sin(A+B) &= 3\sin(A-B) \\
\sin A\cos B + \cos A\sin B &= 3(\sin A\cos B - \cos A\sin B) \\
2\sin A\cos B &= 4\cos A\sin B \\
\tan A &= 2\tan B
\end{aligned}$$

8.

$$\begin{aligned}
\frac{\cos(A+B)+\cos(A-B)}{\sin(A-B)-\sin(A+B)} &= \frac{2\cos A\cos B}{2\cos A\sin(-B)} \\
&= -\cot B
\end{aligned}$$

9. (a)

$$\begin{aligned}
\frac{\sin 5A}{\sin A} - \frac{\cos 5A}{\cos A} &= \frac{\sin 5A\cos A - \sin A\cos 5A}{\sin A\cos A} \\
&= \frac{\sin(5A-A)}{\frac{1}{2}\sin 2A} \\
&= 2\frac{\sin 4A}{\sin 2A} \\
&= 2\frac{2\sin 2A\cos 2A}{\sin 2A} \\
&= 4\cos 2A
\end{aligned}$$

(b) Note that $0^\circ < A < 180^\circ$, so $0^\circ < 2A < 360^\circ$.

$$\begin{aligned}
\frac{\sin 5A}{\sin A} - \frac{\cos 5A}{\cos A} &= 2 \\
4\cos 2A &= 2 \\
\cos 2A &= \frac{1}{2} \\
2A &= 60^\circ \text{ or } 300^\circ \\
A &= 30^\circ \text{ or } 150^\circ
\end{aligned}$$

10. (a)

$$\begin{aligned} \sin 3A &= \sin(2A + A) \\ &= \sin 2A \cos A + \cos 2A \sin A \\ &= 2 \sin A \cos^2 A + \cos 2A \sin A \\ &= 2 \sin A(1 - \sin^2 A) + (1 - 2\sin^2 A) \sin A \\ &= 3 \sin A - 4 \sin^3 A \end{aligned}$$

(b)

$$\begin{aligned} \cos 3A &= \cos(2A + A) \\ &= \cos 2A \cos A - \sin 2A \sin A \\ &= (2\cos^2 A - 1) \cos A - 2 \cos A \sin^2 A \\ &= (2\cos^2 A - 1) \cos A - 2 \cos A(1 - \cos^2 A) \\ &= 4 \cos^2 A - 3 \cos A \end{aligned}$$

(c)

$$\begin{aligned} \tan 3A &= \tan(2A + A) \\ &= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\ &= \frac{(\frac{2\tan A}{1-\tan^2 A}) + \tan A}{1 - (\frac{2\tan A}{1-\tan^2 A}) \tan A} \\ &= \frac{(\frac{3\tan A - \tan^3 A}{1-\tan^2 A})}{(\frac{1-3\tan^2 A}{1-\tan^2 A})} \\ &= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} \end{aligned}$$

11. Note $C = 180° - A - B$, prove that

(a)

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= \sin 2A + \sin 2B + \sin(360° - 2A - 2B) \\ &= \sin 2A + \sin 2B - \sin(2A + 2B) \\ &= \sin 2A + \sin 2B - \sin 2A \cos 2B - \cos 2A \sin 2B \\ &= \sin 2A(1 - \cos 2B) + \sin 2B(1 - \cos 2A) \\ &= 2 \sin A \cos A(2\sin^2 B) + 2 \sin B \cos B(2\sin^2 A) \\ &= 4 \sin A \sin B(\cos A \sin B + \cos B \sin A) \\ &= 4 \sin A \sin B \sin(A + B) \\ &= 4 \sin A \sin B \sin(180° - A - B) \\ &= 4 \sin A \sin B \sin C \end{aligned}$$

(b)

$$
\begin{aligned}
\tan A + \tan B + \tan C &= \tan A + \tan B + \tan(180^\circ - A - B) \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{-\tan^2 A \tan B - \tan A \tan^2 B}{1 - \tan A \tan B} \\
&= \tan A \tan B \frac{-(\tan A + \tan B)}{1 - \tan A \tan B} \\
&= \tan A \tan B (-\tan(A+B)) \\
&= \tan A \tan B \tan(180^\circ - A - B) \\
&= \tan A \tan B \tan C
\end{aligned}
$$

(c)

$$
\begin{aligned}
\cot A \cot B + \cot B \cot C + \cot C \cot A &= \frac{1}{\tan A \tan B} + \frac{1}{\tan B \tan C} + \frac{1}{\tan C \tan A} \\
&= \frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C} \\
&= 1 \qquad \text{(by (b))}
\end{aligned}
$$